

The Power of Discrete Quantum Theories

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We explore the implications of restricting the framework of quantum theory and quantum computation to finite fields. The simplest proposed theory is defined over arbitrary finite fields and loses the notion of unitaries. This makes such theories unnaturally strong, permitting the search of unstructured databases faster than asymptotically possible in conventional quantum computing. The next most general approach chooses finite fields with no solution to $x^2 + 1 = 0$, and thus permits an elegant complex-like representation of the extended field by adjoining $i = \sqrt{-1}$. Quantum theories over these fields retain the notion of unitaries and — for particular problem sizes — allow the same algorithms as conventional quantum theory. These theories, however, still support unnaturally strong computations for certain problem sizes, but the possibility of such phenomena decreases as the size of the field increases.

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Introduction. Quantum computing with complex coefficients technically involves uncomputable numbers and unlimited resources. Specifically, it is well known that the set of computable complex numbers is countable whereas the set of all complex numbers is uncountable. Since we do not completely understand the source of the extended power of conventional quantum computation [1], it is therefore both interesting and potentially important to investigate the possible origins of quantum computational capacity. Here we explore the remarkable properties that result when we replace continuous complex numbers by appropriate finite fields. This apparently simple step adds new and bizarre properties to the well-known post-classical computing power for which quantum computing has justifiably attracted such attention.

We will show that, for finite fields of order p^2 , with the prime p of the form $4r + 3$ (r a non-negative integer), the complex numbers have extremely compelling and natural discrete analogs that permit essentially all of the standard requirements of quantum computing to be preserved. Under suitable conditions, we have amplitude-based partitions of unity, unitary transformations, entanglement, and so forth. What is new is that, because of the cyclic nature of arithmetic in the finite complex field, excessive computational power can result. We explore the mechanisms for these phenomena and speculate on their implications. The circumstances in which such supercomputation can occur depend on special numerical conditions that become more and more scarce as the size of the finite field increases. This leads to the conjecture that, as the size of the field becomes large enough, most of the properties of conventional quantum mechanics would be recovered. This leaves open the question of whether conventional quantum mechanics is physical, or whether perhaps extremely large discrete quantum theories that contain only computable numbers are at the heart of our physical universe.

Modal Quantum Theory. The traditional mathematical framework of conventional quantum theory is that of

Hilbert spaces over the field of complex numbers. Since this field is infinite, it is natural to ask whether versions of quantum theory based on finite fields exist and can be used to approximate conventional quantum theory, thus yielding insights into the power of quantum computing.

Recently Schumacher and Westmoreland [2] showed that it is possible to define versions of quantum theory over finite fields, which they call modal quantum theories. Such theories retain several key quantum characteristics including notions of superposition, interference, entanglement, and mixed states, along with time evolution using invertible linear operators, complementarity of incompatible observables, exclusion of local hidden variable theories, impossibility of cloning quantum states, and the presence of natural counterparts of quantum information protocols such as superdense coding and teleportation. These modal theories are obtained by collapsing the Hilbert space structure over the field of complex numbers to that of a plain vector space over an *arbitrary* finite field. In the resulting structure, all non-zero vectors represent valid quantum states, and the evolution of a closed quantum system is described by *any* invertible linear map.

Specifically, consider a 1-qubit system with basis vectors $|0\rangle$ and $|1\rangle$. In conventional quantum theory, there exist an infinite number of states for a qubit of the form $\alpha|0\rangle + \beta|1\rangle$, with α and β elements of the underlying field of complex numbers subject to the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. Moving to a finite field immediately limits the set of possible states as the coefficients α and β are now drawn from a finite set. In particular, in the field $\mathbb{F}_2 = \{0, 1\}$ of booleans, there are exactly four possible vectors: the zero vector, the vector $|0\rangle$, the vector $|1\rangle$, and the vector $|0\rangle + |1\rangle = |+\rangle$. Since the zero vector is considered non-physical, a 1-qubit system can be in one of only three states. The dynamic evolution of these 1-qubit states is described by any invertible linear map, i.e., by any linear map that is guaranteed never to produce the zero vector from a valid state. There are exactly 6

such maps:

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S^\dagger = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This space of maps is clearly quite impoverished compared to the full set of 1-qubit unitary maps in conventional quantum theory. In particular, it does not include the Hadamard transformation. The space also includes non-unitary maps such as S and S^\dagger that are not allowed in conventional quantum computation.

Measurement in the standard basis is fairly straightforward: measuring $|0\rangle$ or $|1\rangle$ deterministically produces the same state while measuring $|+\rangle$ non-deterministically produces $|0\rangle$ or $|1\rangle$ with no particular probability distribution. In other bases, the measurement process is complicated by the fact that the correspondence between $|\phi\rangle$ and its dual $\langle\phi|$ is basis-dependent and that the underlying finite field is necessarily cyclic. For example, in the field of booleans addition (+) and multiplication (*) are modulo 2, which means that:

$$\langle + | + \rangle = (1 * 1) + (1 * 1) = 1 + 1 = 0. \quad (1)$$

Modal Quantum Computing. Although modal quantum theories were described as “toy” quantum theories, they appear to be endowed with “supernatural” powers. We show next that it is possible — in even the simplest of modal theories — to deterministically solve a black box version of the UNIQUE-SAT problem. The UNIQUE-SAT problem is that of deciding whether a given boolean formula has a satisfying assignment, assuming that it has at most one such assignment. Surprisingly, this problem is, in a precise sense [3], just as hard as the general satisfiability problem and hence all problems in the NP complexity class. Our generalization replaces the boolean formula with an arbitrary classical boolean function. A solution to the generalized problem can be used to solve an unstructured database search of size N using $O(\log N)$ black box evaluations by binary search on the database. This algorithm outperforms the known asymptotic bound $O(\sqrt{N})$ for unstructured database search in conventional quantum computing.

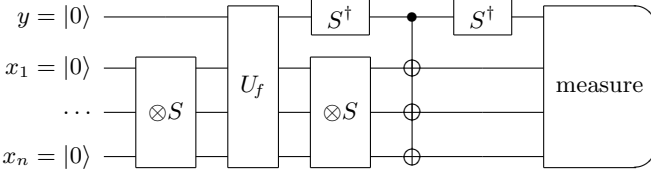


FIG. 1. Circuit for black box UNIQUE-SAT in modal quantum theory over the field \mathbb{F}_2 . For notation see text.

Technically, consider a classical function $f : \text{Bool}^n \rightarrow \text{Bool}$ that takes n bits and that returns at most one **true** result. The algorithm described below (and pictorially in Fig. 1) takes as input such a classical function and decides, deterministically and in a constant number of black box evaluations, whether f is satisfiable or not.

In the following, we use \bar{x} to denote a sequence x_1, x_2, \dots, x_n of n bits. Given the function $f : \text{Bool}^n \rightarrow \text{Bool}$, we construct the Deutsch quantum black box U_f as follows [5]: $U_f |y\rangle |\bar{x}\rangle = |y + f(\bar{x})\rangle |\bar{x}\rangle$. The algorithm consists of the following steps. (1) Initialize an $n+1$ qubit state to $|0\rangle |\bar{0}\rangle$. (2) Apply the map S (defined in the previous section) to each qubit in the second component of the state. (3) Apply the quantum black box U_f to the entire state. (4) Again apply the map S to each qubit in the second component of the state. (5) Apply the map S^\dagger to the first component of the state. (6) Conditional on the first component of the state being $|a\rangle$, apply the map X_a to each qubit in the second component of the state where X_0 and X_1 are defined in the previous section. (7) Again apply the map S^\dagger to the first component of the state. (8) Measure the resulting state in the standard basis for $n+1$ qubits. It is straightforward to calculate that if the measurement yields $|0\rangle |\bar{0}\rangle$ then the function f is *unsatisfiable*. If the measurement is anything else then the function f is *satisfiable*.

Discrete Quantum Theory. We propose variants of modal quantum theories, which we call discrete quantum theories, that aim to exclude “supernatural” algorithms such as the one presented in the previous section by retaining most of the structure of Hilbert spaces over the field of complex numbers. We wish to approximate as closely as possible the following features of conventional quantum theory: (i) the field of complex numbers, (ii) a vector space over the field of complex numbers, and (iii) an inner product $\langle\psi | \phi\rangle$ associating a complex number to each pair of vectors that satisfies the following properties:

- A. $\langle\phi | \psi\rangle$ is the complex conjugate of $\langle\psi | \phi\rangle$;
- B. $\langle\phi | \psi\rangle$ is conjugate linear in its first argument and linear in its second argument;
- C. $\langle\phi | \phi\rangle$ is always non-negative and is equal to 0 only if $|\phi\rangle$ is the zero vector.

Complex Numbers. Although arbitrary finite fields do not have enough structure to represent approximations to the complex numbers, some finite fields do. To understand this point, we review some known facts about finite fields. All finite fields have sizes that are powers of primes. Thus in a field \mathbb{F}_q , the size q must be of the form p^m where the prime number p is known as the *characteristic* of the field and where m is known as the *degree* of the field. The polynomial $x^2 + 1 = 0$ is *irreducible* over a prime field \mathbb{F}_p with p odd if and only if p is of the form $4r+3$, with r a non-negative integer. In other words,

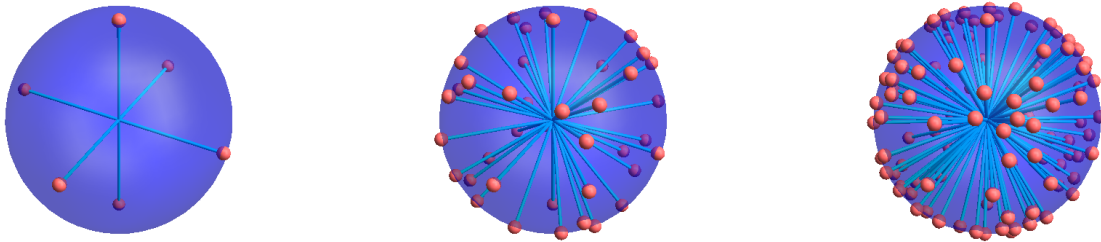


FIG. 2. The discrete versions of the 2-dimensional 1-qubit Hilbert space (the Bloch sphere) that are irreducible distinct state vectors of unit norm in the field, with the finite fields \mathbb{F}_{p^2} for $p = 3$, $p = 7$, and $p = 11$. For example, in \mathbb{F}_{3^2} , there are 24 vectors of norm 1, but only 6 inequivalent vectors, as shown; the 4 equivalent vectors in each class differ only by a discrete phase.

the polynomial is irreducible over $\mathbb{F}_3, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{19}, \dots$ [4]. Furthermore, any such field can be extended to a field \mathbb{F}_{p^2} whose elements can be viewed as complex numbers with the real and imaginary parts in \mathbb{F}_p . In such a special field, the Frobenius automorphism of an element (defined as a^p) agrees with the usual definition of complex conjugation. Concretely, the field \mathbb{F}_{3^2} has 9 elements:

$$0, 1, -1, i, 1+i, -1+i, -i, 1-i, -1-i,$$

which are all the complex numbers one can form using the integers modulo 3 as real and imaginary coefficients. Similarly, the field \mathbb{F}_{7^2} has 49 elements of the form $a+ib$ where a, b are integers in the range $[-3, 3]$ and addition and multiplication are modulo 7.

Inner Products. Consider a d -dimensional vector space over the fields \mathbb{F}_{p^2} that approximate the complex numbers where, in general, there is no connection between the dimension of the vector space d and the characteristic of the field p . Let $|\phi\rangle = (a_0 \ a_1 \ \dots \ a_{d-1})^T$ and $|\psi\rangle = (b_0 \ b_1 \ \dots \ b_{d-1})^T$ with the scalars a_j and b_j drawn from the field elements and where $(.)^T$ is the transpose. The Hermitian dot product of these vectors is:

$$\langle \phi | \psi \rangle = \sum_{j=0}^{d-1} a_j^p b_j$$

This product satisfies conditions A and B for inner products. Condition C is violated in every finite field as there always exists a non-zero vector $|\phi\rangle$ such that $\langle \phi | \phi \rangle = 0$. The reason is that addition in finite fields eventually “wraps around” (because of their cyclic or modular structure) making the notions of positive and negative meaningless and allowing the sum of non-zero elements to be zero. (See Eq. (1) at the end of the first section.) This fact has non-trivial consequences.

Postulates. Given that we can retain most of the structure of conventional quantum theories in finite fields, the postulates of discrete quantum theory below are almost identical to the usual ones. In more detail, the state space of an isolated discrete quantum system is a vector space over a field that approximates the complex

numbers as shown above. In that space, complex conjugates, unitaries, and Hermitian operators have the usual definitions. Furthermore, (1) The state of an isolated system is described by a vector $|\psi\rangle$ such that $\langle \psi | \psi \rangle = 1$, with vectors that differ by a scalar multiplier identified; (2) The state space of a distinguishable composite system is the tensor product of the component systems; (3) Observable quantities are described by operators O such that $O = O^\dagger$; (4) The evolution of the system is described by unitary maps U ; (5) If $|a\rangle$ is an eigenvector of O with eigenvalue a , i.e., if $O|a\rangle = a|a\rangle$, then measurement of property O realizes a . Because of the cyclic nature of the field, traditional probability measures are not directly applicable and some care is needed to define the probabilities of measurement outcomes. We note that our algorithms for solving the black box **UNIQUE-SAT** only rely on probability measures distinguishing certain from impossible events.

Assuming the underlying field to be \mathbb{F}_{3^2} , there are exactly 6 1-qubit state vectors, which appear as symmetric points on the Bloch sphere. We observe that, for \mathbb{F}_{p^2} , there appear to be $p(p-1)$ unique unit norm states on the discrete Bloch sphere, with $(p+1)$ equivalent discrete copies (points on the circle realizing the discrete Hopf fibration) corresponding to each unique state. In Fig. 2 we plot these states on the Bloch sphere for $p = 3, 7$, and 11. Similar discrete maps can be performed for n qubit states.

The unitary operators over the field \mathbb{F}_{3^2} include the Hadamard transformation:

$$H = (1+i) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

whose rows and columns are mutually orthogonal unit vectors. Recall that, in this field, $(1-i)(1+i) = 1+1 = -1$ and that $(-1) + (-1) = 1$ since all operations are modulo 3. No-cloning, the fundamental tenet underlying conventional quantum theory, is respected in our discrete quantum theory.

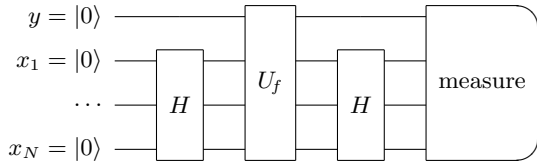


FIG. 3. Circuit for black box **UNIQUE-SAT** in discrete quantum theories.

Any observable O in a 1-qubit space can be written as:

$$O = \sum_{\mu=0}^3 a_{\mu} X_{\mu} = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

where X_0, X_1 were defined above, and the remaining Pauli matrices X_2, X_3 allowed in the field are:

$$X_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $O = O^{\dagger}$ this implies that its diagonal elements can only be 0, 1, -1 , while the off-diagonal ones can be any of the 9 field elements. Incompatibility, i.e., non-commutativity, of observables leads to Heisenberg uncertainty relations among them, a consequence of the validity of Schwarz's inequality in discrete quantum theory.

Discrete Quantum Computing. As shown above, by choosing particular finite fields, it is possible to retain all the structure of conventional quantum theory except for condition C of inner products. The smallest field \mathbb{F}_{3^2} already has enough structure to express the standard Deutsch-Jozsa [5] algorithm as this algorithm only requires normalized versions of vectors or matrices with the scalars 0, 1, and -1 . More complex algorithms such as Grover's database search or Shor's period finding [5] also work as expected in "large enough" finite fields. Consider the diffusion matrix for searching an unstructured database of size N using Grover's algorithm:

$$\begin{pmatrix} -1 + 2/N & 2/N & 2/N & \dots & 2/N \\ 2/N & -1 + 2/N & 2/N & \dots & 2/N \\ 2/N & 2/N & -1 + 2/N & \dots & 2/N \\ \dots & \dots & \dots & \dots & \dots \\ 2/N & 2/N & 2/N & \dots & -1 + 2/N \end{pmatrix}$$

This diffusion matrix is applied \sqrt{N} times. This means that if $N = 4$, the algorithm needs to be expressed using probability amplitudes of the form $\pm \frac{1}{2}$. In the field \mathbb{F}_{3^2} , these amplitudes collapse to ± 1 and the algorithm fails to work. However in the field \mathbb{F}_{7^2} , these amplitudes can properly be expressed and the algorithm works as expected. Generally as the size of the database grows, the size of the underlying field must grow proportionally.

Building a Discrete Quantum Computer. We have argued above that if the field is large enough, then discrete quantum computing approaches conventional quantum computing and that if the size of the field is too small, then the usual algorithms fail to work. However it is possible, in some situations, to exploit the cyclic behavior of the field to creatively cancel probability amplitudes and solve problems with what again appears to be "supernatural" efficiency. We illustrate this behavior with the algorithm in Fig. 3, which is a variant of the algorithm in Fig. 1. However, unlike the situation in modal quantum theories, the algorithm does not always succeed deterministically using a constant number of black box evaluations. This supernatural behavior only happens if the characteristic p of the field divides $2^N - 1$. For a database of fixed size, this match becomes less likely as the size of the field increases. For a given field, it is possible to expand any database with dummy records to satisfy the divisibility property.

Physical connections. We conclude by pointing out a connection between our discrete quantum theories and Schwinger's foundational attempts to formulate quantum mechanics from measurement [6, 7]. He proposed a measurement algebra derived from a selected set of experiments, including coordinates and momenta whose eigenvalues are modular integers. Interestingly, although this formulation shares with the discrete quantum theories the cyclic structure induced by the finite fields, it differs in that the infinite complex numbers are used to define a state space that is a Hilbert space with a standard inner product. Related situations also appear in quantum field theories, where competition between confinement and deconfinement of elementary particles may appear depending upon the compact (*cyclic*) or non-compact nature of the quantum fields.

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- [1] We use the phrase "conventional quantum theory" where necessary to distinguish the usual quantum theory and quantum computing paradigm using (continuous) complex numbers from discrete quantum theory. Alternative terminology in the literature includes "actual," "standard," and "ordinary" quantum theory.
 - [2] B. Schumacher and M. D. Westmoreland, *Workshop on Quantum Physics and Logic*, B. Coecke, P. Panangaden, and P. Selinger eds., 2010, pp. 145–149.
 - [3] L. G. Valiant and V. V. Vazirani, *Theor. Comput. Sci.*, **47**:85 – 93, 1986.
 - [4] Fields \mathbb{F}_q where q is a power of a prime p , i.e., $q = p^m$, are known as Galois fields.
 - [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
 - [6] J. Schwinger, *Quantum Mechanics: Symbolism of Atomic Measurements* (Springer Verlag, Berlin, 2001).
 - [7] A. Vourdas, *J. Phys. A: Math. Theor.* **40** R285 (2007).